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On a theorem on partially summing tangles by Lickorish

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Kirby and Lickorish have introduced the idea of prime tangle (5) where the term 'tangle' is borrowed from Conway (3). While exploring this notion, Lickorish has shown in ((6), theorem 3): A tangle, which is the Conway sum of two tangles, one of which is prime and connected to both arcs of the other one which itself is prime or untangle, is prime.

In this note, we generalize this result and give therefore the necessary and sufficient condition for a tangle which results from the Conway sum of two tangles, to be prime. The proof given here is elementary in the sense that it avoids the two-fold branched cover.

Definition. A *tangle* is a pair (B, t) where B is a 3-ball and t is a set of two disjoint arcs embedded in B with $t \cap \partial B = \partial t$.

Definition. A tangle (B, t) is *untangled* or *rational* if the two arcs of t are unknotted and if there exists a properly embedded disc D in B, i.e. $D \cap \partial B = \partial D$, which separates the two arcs.

Every untangle can be constructed in the following way: starting from the corner points of a square 'pillowcase', draw p/q slope lines on its top and p/q slope lines on its bottom for some $p/q \in \mathbb{Q} \cup \{1/0\}$; we shall therefore denote an untangle by $T_{p/q}$ (see Fig. 1).

Given two tangles (B_1, t_1) and (B_2, t_2), a tangle obtained by identifying a (disc, point pair of ∂t_1) on ∂B_1 to a (disc, point pair of ∂t_2) on ∂B_2 is called a *partial sum* of (B_1, t_1) and (B_2, t_2). A partial sum of two tangles (B_1, t_1) and (B_2, t_2), with the choice of identification depicted in Fig. 2, is called the *Conway sum* of (B_1, t_1) and (B_2, t_2). It will be denoted by $(B_1, t_1) + (B_2, t_2)$.

Let $T = (B, t)$ be a tangle. Depicted in Fig. 3 are two ways to add the untangle which complete the arcs t into a single or two-component link. They are respectively called the denominator $D(T)$ and the numerator $N(T)$ of the tangle T. Note that the string orientation is not taken into consideration here.

For a rational tangle $T_{p/q}$, the denominator $D(T_{p/q})$ and the numerator $N(T_{p/q})$ produce respectively the two-bridge knot or link $K_{p/q}$ and $K_{-q/p}$. Depending on the parity of q, $K_{p/q}$ is a knot (for q odd) or a link (for q even).

Definition. A tangle (B, t) is *prime* if it satisfies the following conditions:

(i) Any 2-sphere in B, which meets t transversely in two points, bounds in B a ball meeting t in an unknotted spanning arc.

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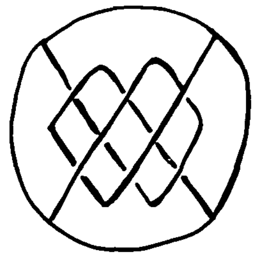
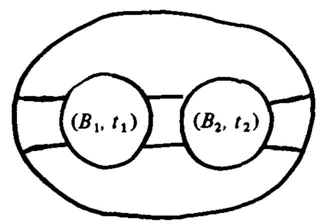


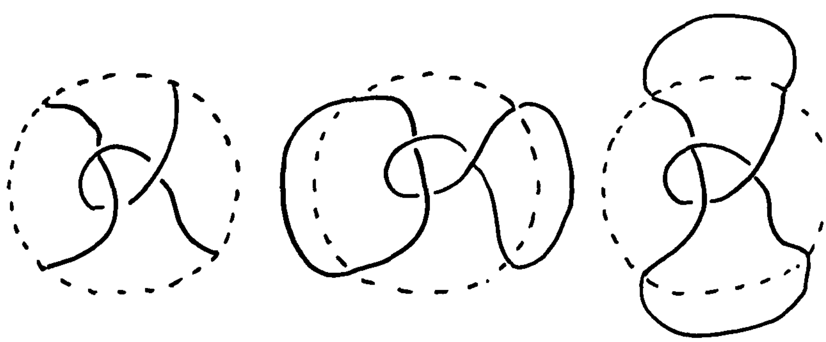
$T_{\frac{3}{2}}$

Fig. 1.



$(B_1, t_1) + (B_2, t_2)$

Fig. 2.



$T = (B, t)$ $D(T)$ $N(T)$

Fig. 3.

(ii) The arcs of t cannot be separated by a disc properly embedded in B.
The condition (ii), in the presence of (i), is equivalent to the following condition:

(ii′) (B, t) is not an untangle.

Let (C, v) be a tangle which results from the Conway sum of two tangles such that one is the tangle (A, α) and the other is a rational tangle $T_{p/q}$ (Fig. 4).

LEMMA 1. *(C, v) is a prime tangle if $p/q \neq 1/0$.*

Proof. q odd. Both arcs of $T_{p/q}$ are thus connected to the tangle (A, α). Since (A, α) is obviously prime, it follows from ((6), theorem 3) that (C, v) is also prime.

q even. Each of the arcs of (C, v) is unknotted. Thus, (C, v) contains no knotted arc-ball pair. (C, v) is not an untangle as the numerator of (C, v) produces a non-simple knot, i.e. a knot with no trivial companion. Indeed, by (4) a two-bridge knot is simple. ∎

This lemma gives an elementary proof to the following propositions which avoids the resource of the two-fold branched cover.

Proposition 2. *Let (F,f) be a tangle which can be expressed as the Conway sum of two rational tangles T_{p_1/q_1} and T_{p_2/q_2} with $p_i/q_i \in \mathbb{Q}-\mathbb{Z}$* for $i = 1,2$. *Then (F,f) is a prime tangle.*

Proof. Consider two copies (A_1, α_1) and (A_2, α_2) of the tangle (A, α) depicted in Fig. 4. Perform a partial sum of (F,f) with these two copies such that the resulting tangle, denoted by (S, l), is shown on Fig. 5.

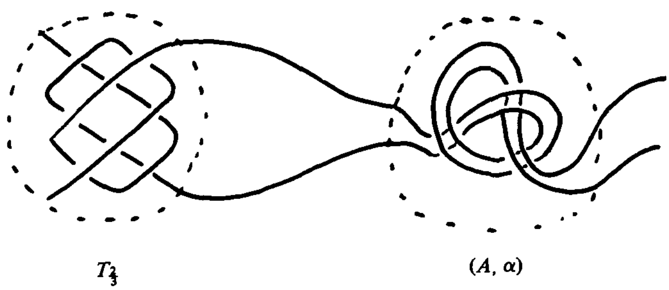


Fig. 4.

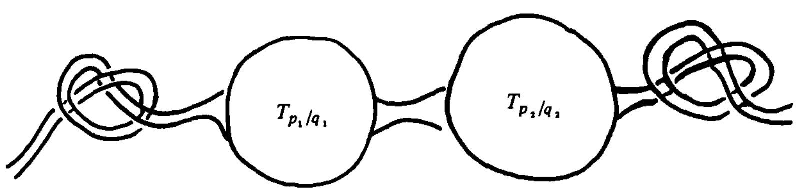


Fig. 5.

By Lemma 1, with the hypothesis $p_i/q_i \neq 1/0$ for $i = 1,2$, the tangles $\{(A_1, \alpha_1) + T_{p_1/q_1}\}$ and $\{T_{p_2/q_2} + (A_2, \alpha_2)\}$ are both prime. Since (S, l) is the Conway sum of these two tangles, (S, l) is also prime by ((6), theorem 3). It then follows that the tangle (F,f) embedded in (S, l) inherits from (S, l) the property to be without knotted arc-ball pair. In our assumptions on p_i/q_i, when one forms the denominator of (F,f), one obtains a connected sum of two-bridge knots or links. Thus, (F,f) is not an untangle. ∎

In ((2) and references therein), this kind of prime tangle which is a Conway sum of two rational tangles, has been considered in the context of algebraic (or plumbing type) knots.

Proposition 3. *Let (G, g) be a tangle which results from the Conway sum of two tangles. Suppose that one, denoted by (B_1, s_1), is prime and the other, denoted by (B_2, s_2), is either prime or untangled $T_{p/q}$ with $p/q \neq 1/0$. Then (G, g) is prime.*

Proof. The case where (B_2, s_2) is prime or rational $T_{p/q}$ with q odd, i.e. both arcs are connected to (B_1, s_1), is already given by ((6), theorem 3). Therefore, only the case where (B_2, s_2) is a rational tangle $T_{p/q}$ with $p/q \in \mathbb{Q}-\mathbb{Z}$ remains. Perform the Conway sum of (G, g) with a rational tangle $T_{p'/q'}$ with q' odd and $p'/q' \in \mathbb{Q}-\mathbb{Z}$. Denote the resulting tangle by (S', l'). Since $p/q \in \mathbb{Q}-\mathbb{Z}$ and $p'/q' \in \mathbb{Q}-\mathbb{Z}$ by assumption, $\{T_{p/q} + T_{p'/q'}\}$ is prime by Proposition 2. It then follows that (S', l'), which can be expressed as the Conway sum of (B_1, s_1) and $\{T_{p/q} + T_{p'/q'}\}$, is also prime by ((6), theorem 3). The tangle (G, g) embedded in (S', l') is therefore without a knotted arc-ball pair.

If (G, g) were a rational tangle, the numerator of (S', l') would produce a two-bridge knot K. However, since $\{T_{p/q} + T_{p'/q'}\}$ and (B_1, s_1) are prime, K would be expressed as a result of a sum of prime tangles (given two tangles and a homeomorphism of $f: (\partial C_1, \partial t_1) \to (\partial C_2, \partial t_2)$, the result $(C_1, t_1) \cup_f (C_2, t_2)$ is referred as a sum of two tangles). This would contradict the known fact that a two-bridge knot is of tunnel number one and thus, could not result from a sum of two prime tangles ((7), (1)). ∎

THEOREM 4. *Let (G, g) be a tangle which can be expressed as the Conway sum of two tangles (C_1, t_1) and (C_2, t_2). The tangle (G, g) is prime if and only if one of the following conditions hold:*

(1) *One of the two tangles (C_1, t_1) and (C_2, t_2) is prime and the other is either prime or rational $T_{p/q}$ with $p/q \neq 1/0$.*

(2) *(C_1, t_1) and (C_2, t_2) are both untangles T_{p_1/q_1} and T_{p_2/q_2} with $p_i/q_i \in \mathbb{Q} - \mathbb{Z}$ for $i = 1, 2$.*

Proof. The sufficiency is given by Propositions 2 and 3. As for necessity, it is obvious that (C_i, t_i) has to be either prime or untangled.

If one of the two tangles (C_1, t_1) and (C_2, t_2) is the rational tangle $T_{1/0}$, one can find obviously a properly embedded disc in G which separates the two arcs of (G, g).

If (C_i, t_i) is T_{p_i/q_i} for $i = 1, 2$, with at least one $q_i = 1$, for instance q_2, $\{T_{p_1/q_1} + T_{p_2}\}$ results in the rational tangle $T_{(p_1+p_2 q_1)/q_1}$. ∎

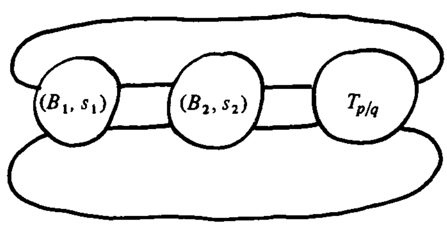


Fig. 6.

Let L be a single or two component link in S^3 which can result from a sum of tangles depicted in Fig. 6, where (B_1, s_1) and (B_2, s_2) are prime and (B_3, s_3) is a rational tangle $T_{p/q}$.

COROLLARY 5. *If $p/q \neq 1/0$, L is prime.*

Proof. This is a consequence of Theorem 4 and ((6), theorem 1).

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